Chapter 6

Powerspaces, multifunctions and predicate transformers

Programming languages can often be defined in terms of atomic statements (like assignments to variables), a set of statement operators (like sequential composition and non-deterministic choice), a set of process variables, and a recursion operator for each process variable. To give a compositional semantics to a program it is therefore necessary to define a semantic domain in which atomic statements and statements operators can be interpreted. Modeling recursion is one of the difficult aspects of building a compositional semantics. For this reason the input and output state spaces of a program are often structured, complete with respect to some limit construction, and recursively defined.

In Chapter 3 we introduced two different models for a compositional semantics of a programming language: the state transformer model and the predicate transformer model. A rich collection of semantic constructions is available for state transformers on structured sets of states. However, the same cannot be said for predicate transformers. Nothing, or very little, is known about compositional predicate transformer semantics for programs which interact with their environment.

In this chapter we investigate the relationships between state transformers and predicate transformers in a general topological setting. Topological dualities between predicate and state transformers provide a mathematical approach to predicate transformers between structured sets of states. The connection between state transformers and topological predicate transformers was first studied by Smyth [179] who placed the result of Plotkin [159] for the Smyth powerdomain in a broader topological framework using the upper powerspace. This work is our starting point. However Smyth restricts to sober spaces while we use $\mathcal{T}_0$ spaces. Also, our techniques are more in line with the ones used in [159]. Besides the upper powerspace we consider also the lower and the Vietoris powerspaces, and we show that the three isomorphisms established in Chapter 3 also hold in this general setting. In passing, topological representations of order theoretic and metric powerdomains are given.

All topological dualities we describe are order-preserving. As a consequence, to define a predicate transformer semantics from a state transformer semantics it is sufficient to define only predicate transformers for the atomic statements and operators on predicate transformers corresponding to syntactical operators. Recursive constructs can be handled in the predicate transformer semantics exactly in the same way as for the state transformer semantics.

6.1 Multifunctions as state transformers

One way to capture a compositional semantics of a concurrent program is to consider it as a function from input states to the set of all intermediate states through which the program P passes after one atomic step, followed by the semantics of the remaining part of the program to be executed. In order to deal with this recursive definition, states are usually endowed with a topological structure (usually a partial order or a distance function). To model non-deterministic computations, semantic functions can be represented by many-valued functions, or multifunctions.

Definition 6.1.1 *A* multifunction $f : X \to Y$ *with values in* $\mathcal{V} \subseteq \mathcal{P}(Y)$ *is a function that assigns to every element* x *of a topological space* X *a subset* $f(x) \in \mathcal{V}$ *of a topological space* Y*.*

For a multifunction $f : X \to Y$ and a predicate P on Y, we denote by $f^+(P)$ the *upper inverse* of f, that is, the set of all inputs x of f such that every element of $f(x)$ satisfies the predicate P. The *lower inverse* of f is denoted by $f^-(P)$ and is defined as the set of all inputs x of f such that some elements of $f(x)$ satisfy the predicate P. Formally, for $P \subseteq Y$:

$$f^+(P) = \{x \in X \mid f(x) \subseteq P\} \text{ and } f^-(P) = \{x \in X \mid f(x) \cap P \neq \emptyset\}.$$

The maps f^+ and f^- are *dual* in the sense that $f^+(P) = X \setminus f^-(Y \setminus P)$ for all $P \subseteq Y$. Different ways of defining inverse give rise to different ways of defining continuity. Below we list three definitions of continuity for a multifunction [27].

Definition 6.1.2 *Let X and Y be two topological spaces. A multifunction $f : X \to Y$ with values in $\mathcal{V} \subseteq \mathcal{P}(Y)$ is said to be*

(i) lower semi-continuous *if $f^-(o) \in \mathcal{O}(X)$ for every $o \in \mathcal{O}(Y)$,*

(ii) upper semi-continuous *if $f^+(o) \in \mathcal{O}(X)$ for every $o \in \mathcal{O}(Y)$, and*

(iii) continuous *if it is both upper and lower semi-continuous.*

Since f^- and f^+ are dual functions, the above notions of continuity could also have been expressed in terms of refutative predicates rather than affirmative ones. For example, f is lower semi-continuous if and only if $f^+(c)$ is a closed subset of X for every closed subset c of Y. For every notion of continuity of a multifunction there is a related topology on the collection of subsets of the codomain [141,153].

Definition 6.1.3 *Let $\mathcal{V}$ be a set of subsets of a space X.*

(i) *The* lower topology *on $\mathcal{V}$ has as sub-base the collection of all sets of the form L_o for $o \in \mathcal{O}(X)$, where*

$$L_o = \{S \in \mathcal{V} \mid S \cap o \neq \emptyset\}.$$

(ii) *The* upper topology *on $\mathcal{V}$ is defined by taking as base the the collection of all sets of the form U_o for $o \in \mathcal{O}(X)$, where*

$$U_o = \{S \in \mathcal{V} \mid S \subseteq o\}.$$

(iii) *The* Vietoris topology *on $\mathcal{V}$ has as sub-base the union of the base of the upper topology and the sub-base of the lower topology.*

The definitions of the (sub-)bases of the above topologies are chosen in this way in order to make the proof of the following proposition trivial [141] (see also [179]).

Proposition 6.1.4 *Let X, Y be two topological spaces, and let $f : X \to Y$ be a multifunction with values in $\mathcal{V} \subseteq \mathcal{P}(Y)$ with $\mathcal{V} \neq \emptyset$. Then*

(i) *$f : X \to Y$ is lower semi-continuous if and only if the corresponding function $f : X \to \mathcal{V}$ is continuous with respect to the lower topology on $\mathcal{V}$;*

(ii) *$f : X \to Y$ is upper semi-continuous if and only if $f : X \to \mathcal{V}$ is continuous with respect to the upper topology on $\mathcal{V}$; and*

(iii) *$f : X \to Y$ is continuous if and only if $f : X \to \mathcal{V}$ is continuous with respect to the Vietoris topology on $\mathcal{V}$.*

Moreover, these three topologies on $\mathcal{V}$ are the only ones that have these properties. $\square$

In general, for an arbitrary collection of subsets $\mathcal{V}$ of a space X, the upper, lower and Vietoris topologies on $\mathcal{V}$ do not ensure that the resulting space is $\mathcal{T}_0$.

Lemma 6.1.5 *Let $\mathcal{V}$ be a set of subsets of a space X, and $A, B \in \mathcal{V}$,*

(i) *$A \precsim B$ in the preorder induced by the lower topology on $\mathcal{V}$ if and only if $cl(A) \subseteq cl(B)$, where cl is the closure operator induced by the topology on X;*

(ii) *$A \precsim B$ in the preorder induced by the upper topology on $\mathcal{V}$ if and only if $\uparrow A \supseteq \uparrow B$, where the upper closure is taken with respect to the specialization preorder of X;*

(iii) *$A \precsim B$ in the preorder induced by the Vietoris topology on $\mathcal{V}$ if and only if both $cl(A) \subseteq cl(B)$ and $\uparrow A \supseteq \uparrow B$.*

Proof. (i) Let $o \in \mathcal{O}(X)$ be such that $A \in L_o$. Since $A \subseteq cl(A)$, $cl(A) \cap o \neq \emptyset$. If $cl(A) \subseteq cl(B)$ then also $cl(B) \cap o \neq \emptyset$. It follows that also $B \cap o \neq \emptyset$, because otherwise $B \subseteq X \setminus o$ would imply $cl(B) \subseteq X \setminus o$, contradicting $B \cap o \neq \emptyset$. Hence $B \in L_o$. For the converse, assume $A \in L_o$ implies $B \in L_o$ for every open o. Since $B \subseteq cl(B)$, $B \notin L_{X \setminus cl(B)}$. Hence also $A \notin L_{X \setminus cl(B)}$, that is, $A \subseteq cl(B)$. Therefore $cl(A) \subseteq cl(B)$.

(ii) Assume $A \in U_o$ for some $o \in \mathcal{O}(X)$. Then $\uparrow A \subseteq o$ by definition of specialization preorder. If $\uparrow A \supseteq \uparrow B$, then also $\uparrow B \subseteq o$. But $B \subseteq \uparrow B$, thus $B \in U_o$. Conversely, assume $A \in U_o$ implies $B \in U_o$ for every open o. Since $\uparrow A = \bigcap\{o \in \mathcal{O}(X) \mid A \subseteq o\}$, we can immediately conclude that $\uparrow A \supseteq \uparrow B$.

(iii) Combine the two previous items. $\square$

The above lemma justifies the following restriction which considers only certain kinds of subsets of a space. Starting from a topological space X, we consider three spaces of subsets of X:

(i) the *lower powerspace* of X, denoted by $\mathcal{P}_l(X)$ and defined as the collection of all closed subsets of X taken with the lower topology;

(ii) the *upper powerspace* of X, denoted by $\mathcal{P}_u(X)$ and defined as the collection of all upper closed subsets of X taken with the upper topology; and

(iii) the *convex powerspace* of X, denoted by $\mathcal{P}_c(X)$ and defined as the collection of all convex closed subsets of X taken with the Vietoris topology, where $S \subseteq X$ is convex closed if $S = cl(S) \cap \uparrow S$.

Variations of the above powerspaces can be obtained by deleting the empty set or restricting to finitarily specifiable subsets using compact sets. Below we denote by $\mathcal{P}_u^{co}(X)$ the collection of all upper closed and compact subsets of X taken with the upper topology, whereas $\mathcal{P}_c^{co}(X)$ denotes the collection of all convex closed and compact subsets of X taken with the Vietoris topology.

From Lemma 6.1.5 it follows that the three powerspaces above are $\mathcal{T}_0$ (more precisely, they are isomorphic in **Sp** to the $\mathcal{T}_0$-ification of $\mathcal{P}(X)$ taken with the lower, upper and Vietoris topology, respectively).

Let X and Y be two topological spaces. Three posets of *topological state transformers* can be identified:

- the *lower state transformers*, i.e. continuous functions from X to $\mathcal{P}_l(Y)$ ordered by the pointwise extension of the specialization preorder induced by the lower topology;
- the *upper state transformers*, i.e. continuous functions from X to $\mathcal{P}_u(Y)$ ordered by the pointwise extension of the specialization preorder induced by the upper topology;
- the *convex state transformers*, i.e. continuous functions from X to $\mathcal{P}_c(Y)$ ordered by the pointwise extension of the specialization preorder induced by the Vietoris topology.

The above domains of topological state transformers can be related to the three domains of state transformers introduced in Chapter 3 as follows. Let X, Y be two sets, and consider the flat cpo $Y_\perp$ taken with the Scott topology. Then, by definition of the Scott topology on $Y_\perp$,

$$\begin{aligned}
\mathcal{P}_l(Y_\perp) &= \{S \subseteq Y \cup \{\perp\} \mid S \neq \emptyset \Rightarrow \perp \in S\},\\
\mathcal{P}_u(Y_\perp) &= \mathcal{P}(Y) \cup \{Y_\perp\},\\
\mathcal{P}_c(Y_\perp) &= \mathcal{P}(Y \cup \{\perp\}).
\end{aligned}$$

Hence $\mathcal{P}_l(Y_\perp) \setminus \{\emptyset\} \cong \mathcal{P}(Y)$. If we take X with the discrete topology then every function from X to one of the three powerspaces above is continuous.

By the Definitions 3.2.6, 3.2.1 and 3.2.9 of the Hoare, Smyth, and Egli-Milner state transformers, respectively, it follows that

$$\begin{aligned} ST_H(X, Y) &\cong X \to (\mathcal{P}_l(Y_\perp) \setminus \{\emptyset\}), \\ ST_S(X, Y) &= X \to \mathcal{P}_u(Y_\perp), \text{ and} \\ ST_E(X, Y) &= X \to \mathcal{P}_c(Y_\perp). \end{aligned}$$

A subset of $Y_\perp$ is compact in the Scott topology if and only if it is either finite or contains the bottom element $\perp$. Therefore

$$ST_S^{fin}(X, Y) = X \to \mathcal{P}_u^{co}(Y_\perp) \text{ and } ST_E^{fin}(X, Y) = X \to \mathcal{P}_c^{co}(Y_\perp).$$

In the next section we relate the lower, the upper and the convex state transformers with predicate transformers between affirmative predicates.

6.2 Topological predicate transformers

Since affirmative predicates are identified with the open sets of a topological space, functions from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$ are the appropriate *topological* generalization of predicate transformers. For ordinary predicate transformers, *complete multiplicativity* (preservation of arbitrary meets) is required to rule out those predicate transformers which represent 'imaginary programs' (specifications). In addition, Scott continuity is required on predicate transformers to characterize 'computable' programs. While the latter constraint can be easily exported to our topological generalization of predicate transformers (open sets are closed under arbitrary unions), the condition of complete multiplicativity requires more attention: open sets are not closed under arbitrary intersections.

Definition 6.2.1 *Let X and Y be two topological spaces. A function π from the lattice of opens $\mathcal{O}(Y)$ to the lattice of opens $\mathcal{O}(X)$ is said to be* M-multiplicative *if whenever $\bigcap P \subseteq \bigcap Q$ then also $\bigcap \pi(P) \subseteq \bigcap \pi(Q)$, for all $P, Q \subseteq \mathcal{O}(Y)$. The collection of all M-multiplicative functions from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$ is denoted by $\mathcal{O}(Y) \to_M \mathcal{O}(X)$.*

Intuitively, an M-multiplicative predicate transformer preserves specifications: if a specification Q on the output space of some program (denoted by an M-multiplicative predicate transformer π) is refined by another specification P, then every input x which makes the output of the program satisfy P should also make the the output of the program satisfy Q.

One can easily verify that M-multiplicative functions are monotone with respect to subset inclusion. Moreover, they preserve all intersections of open sets which are open. Since in every space the empty intersection is the top element in the lattice of open sets, M-multiplicative functions are top-preserving. For M-multiplicative functions we can prove the following *stability lemma* which generalizes Lemma 3.3.5.

Lemma 6.2.2 *Given two spaces X and Y, let $\pi : \mathcal{O}(Y) \to_M \mathcal{O}(X)$ be an M-multiplicative function. Then*

$$x \in \pi(u) \textit{ if and only if } \bigcap\{o \in \mathcal{O}(Y) \mid x \in \pi(o)\} \subseteq u$$

for every u open in Y and $x \in X$.

Proof. The direction from left to right is obvious. For the converse we use M-multiplicativity: if $\bigcap\{o \in \mathcal{O}(Y) \mid x \in \pi(o)\} \subseteq u$ then $\bigcap\{\pi(o) \mid x \in \pi(o)\} \subseteq \pi(u)$. Hence $x \in \pi(u)$. $\square$

The M-multiplicative functions arise naturally from upper semi-continuous multifunctions. If $f : X \to Y$ is an upper semi-continuous multifunction, then its upper inverse $f^+ : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an M-multiplicative predicate transformer. Assume $\bigcap P \subseteq \bigcap Q$ for P and Q arbitrary collections of opens of Y, and let $x \in \bigcap\{f^+(o) \mid o \in P\}$. Then $f(x) \subseteq o$ for all $o \in P$ and hence $f(x) \subseteq o$ for all $o \in Q$. Therefore $x \in \bigcap\{f^+(o) \mid o \in Q\}$, which proves f^+ is M-multiplicative.

Dually, if $f : X \to Y$ is a lower semi-continuous multifunction then its lower inverse $f^- : \mathcal{O}(Y) \to \mathcal{O}(X)$ preserves all unions, that is, f^- is *completely additive*. The collection of all completely additive functions from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$ is denoted by $\mathcal{O}(Y) \to_A \mathcal{O}(X)$. For completely additive functions we have the following stability lemma.

Lemma 6.2.3 *Given two spaces X and Y, let $\pi : \mathcal{O}(Y) \to_A \mathcal{O}(X)$ be a completely additive function. Then*

$$x \notin \pi(u) \textit{ if and only if } u \subseteq \bigcup\{o \in \mathcal{O}(Y) \mid x \notin \pi(o)\}$$

for every u open in Y and $x \in X$.

Proof. The direction from left to right is obvious. Conversely, if $u \subseteq \bigcup\{o \in \mathcal{O}(Y) \mid x \notin \pi(o)\}$ then $\pi(u) \subseteq \bigcup\{\pi(o) \mid x \notin \pi(o)\}$ because π is completely additive (and hence also monotone). Therefore $x \notin \pi(u)$. $\square$

Notice that the completely additive functions are about refutative predicates. Next we provide duality results between state transformers and topological predicate transformers. They extend the results of Chapter 3 to arbitrary topological spaces. Both the order-theoretic and the metric state transformers are instances of topological state transformers. Since there is a rich semantical theory for order-based state transformers as well as for metric-based state transformers, the dualities below give an indirect way to define predicate transformer semantics for programming languages.

Lower state transformers

Lower state transformers are related to completely additive predicate transformers. The isomorphism below can be used to give a semantic interpretation of one domain in terms of the other. The mapping γ from state transformers to predicate transformers explains that lower state transformers model nondeterministic computations which 'may' satisfy an affirmative predicate. Conversely, the map γ^{-1} from predicate transformers to state transformers tells us that completely additive predicate transformers are about safety: a state x satisfies $\pi(P)$ if the computation represented by π at input x is guaranteed not to terminate in a state not satisfying the affirmative predicate P.

Theorem 6.2.4 *Let X and Y be two topological spaces. The poset of lower state transformers $X \to \mathcal{P}_l(Y)$ is order isomorphic to the poset of completely additive functions $\mathcal{O}(Y) \to_A \mathcal{O}(X)$.*

Proof. We use Proposition 6.1.4. For every continuous function $f : X \to \mathcal{P}_l(Y)$ and completely additive predicate transformer $\pi : \mathcal{O}(Y) \to_A \mathcal{O}(X)$ define the maps $f \mapsto \gamma(f)$ and $\pi \mapsto \gamma^{-1}(\pi)$ by

$$\begin{aligned} \gamma(f) &= \lambda o \in \mathcal{O}(Y).\{x \in X \mid f(x) \cap o \neq \emptyset\} \qquad \text{and} \\ \gamma^{-1}(\pi) &= \lambda x \in X.\, Y \setminus \bigcup\{o \in \mathcal{O}(Y) \mid x \notin \pi(o)\}. \end{aligned}$$

First note that $\gamma(f)(o) = f^{-}(o)$. Hence $\gamma(f)$ is completely additive and, because f is lower semi-continuous as multifunction, well-defined. To prove that $\gamma^{-1}(\pi)$ is lower semi-continuous we see that clearly $\gamma^{-1}(\pi)(x)$ is a closed subset of Y for every $x \in X$, and moreover, for every $o \in \mathcal{O}(Y)$,

$$\begin{aligned}
(\gamma^{-1}(\pi))^{-}(o) &= \{x \in X \mid \gamma^{-1}(\pi)(x) \cap o \neq \emptyset\} \\
&= \{x \in X \mid o \not\subseteq \bigcup\{u \in \mathcal{O}(Y) \mid x \notin \pi(u)\} \\
&= \{x \in X \mid x \in \pi(o)\} \quad \text{[Lemma 6.2.3]} \\
&= \pi(o).
\end{aligned}$$

Since $\pi(o)$ is open in X, $\gamma^{-1}(\pi)$ is lower semi-continuous. Thus it is well-defined. The above also proves that γ^{-1} is a right inverse of γ. It is also a left inverse because, for every $x \in X$,

$$\begin{aligned}
\gamma^{-1}(\gamma(f))(x) &= Y \setminus \bigcup\{o \in \mathcal{O}(Y) \mid x \notin \gamma(f)(o)\} \\
&= Y \setminus \bigcup\{o \in \mathcal{O}(Y) \mid f(x) \cap o = \emptyset\} \\
&= \bigcap\{c \in \mathcal{C}(Y) \mid f(x) \subseteq c\} \\
&= f(x),
\end{aligned}$$

where the latter equality follows because $f(x)$ is closed in Y. Preservation of orders is immediate. $\square$

If a continuous function $f : X \to \mathcal{P}_l(Y)$ is non-empty for all $x \in X$, then $\gamma(f)$ is strict, whereas, if $\pi : \mathcal{O}(Y) \to_A \mathcal{O}(X)$ is a strict a completely additive predicate transformer then $\gamma^{-1}(\pi)(x) \neq \emptyset$ for all $x \in X$.

The following corollary restricts the above duality to a finitary one for locally open compact spaces.

Corollary 6.2.5 *Let X and Y be two locally open compact spaces. The poset of lower state transformers $X \to \mathcal{P}_l(Y)$ is order isomorphic to the poset of finitely additive functions in $\mathcal{KO}(Y) \to \mathcal{KO}(X)$.*

Proof. Since X and Y are locally open compact, the collections of their compact open sets form bases for their respective topologies. Moreover, because a function preserves all joins if and only if it preserves all the directed joins and the finite ones, the order isomorphism of Theorem 6.2.4 cuts down to an order isomorphism between $X \to \mathcal{P}_l(Y)$ and the finite unions preserving functions in $\mathcal{KO}(Y) \to \mathcal{KO}(X)$. $\square$

A natural question is whether the locally open compact spaces are closed under the lower powerspace construction. The answer is given in the following proposition which is similar to Proposition 6.11 in [170].

Proposition 6.2.6 *If X is a locally open compact space then so is $\mathcal{P}_l(X)$.*

Proof. Let X be a locally open compact space, and let $A \in \mathcal{P}_l(X)$ be such that $A \in L_{o_1} \cap ... \cap L_{o_n}$, where all o_i's are open subsets of X. In order to show that $\mathcal{P}_l(X)$ is also locally open compact we have to find an open compact set of $\mathcal{P}_l(X)$ containing A as element and contained in $L_{o_1} \cap ... \cap L_{o_n}$. Since $A \in L_{o_1} \cap ... \cap L_{o_n}$ we can find $x_i \in A \cap o_i$. By locally open compactness of X we can therefore find compact open subsets u_i of X such that $x_i \in u_i \subseteq o_i$ for all $i \in \{1, ..., n\}$. Consider the open set $L_{u_1} \cap ... \cap L_{u_n}$ of $\mathcal{P}_l(X)$. By construction $A \in L_{u_1} \cap ... \cap L_{u_n} \subseteq L_{o_1} \cap ... \cap L_{o_n}$. It remains to prove the compactness of $L_{u_1} \cap ... \cap L_{u_n}$. Using Proposition 5.3.4 (Alexander sub-basis theorem) it is enough to find a finite subset $K \subseteq J$ for every index set J such that $L_{u_1} \cap ... \cap L_{u_n} \subseteq \bigcup_J L_{o_j}$, where all o_j's are open subsets in X. If $L_{u_1} \cap ... \cap L_{u_n} \subseteq \bigcup_J L_{o_j}$ then $u_1 \cup ... \cup u_n \subseteq \bigcup_J o_j$. Hence, by compactness of u_i it follows that there exists a finite index set $K \subseteq J$ such that $u_1 \cup ... \cup u_n \subseteq \bigcup_K o_j$. Therefore $L_{u_1} \cap ... \cap L_{u_n} \subseteq \bigcup_K L_{o_j}$, from which the required compactness follows. $\square$

Closure properties of the lower space construction have been extensively studied by Schalk in her thesis [170]. Using the lower powerlocale as defined in [166], Schalk [170, Proposition 6.26] proved that sober spaces are closed under the lower space construction. Algebraic cpo's taken with the Scott topology are sober and locally open compact (see Chapter 5). What is the connection between the lower space and the Hoare powerdomain of an algebraic cpo? For ω-algebraic cpo's the question has been answered by Smyth [179], whereas for more general (continuous) domains the answer can be found in [170,4,146].

Proposition 6.2.7 *For an algebraic cpo X, the Hoare powerdomain $\mathcal{H}(X)$ taken with the Scott topology is isomorphic in* **Sp** *to the non-empty lower space $\mathcal{P}_l(X) \setminus \{\emptyset\}$.* $\square$

Since continuous functions between two algebraic cpo's X and Y with the Scott topology are exactly the functions preserving the least upper bounds of directed sets, from Theorem 6.2.4 and the above proposition we have the following duality. The poset of Scott continuous functions $X \to \mathcal{H}(Y)$ is order isomorphic to the poset of all strict and completely additive functions from the lattice of Scott opens $\mathcal{O}(Y)$ to the lattice of Scott opens $\mathcal{O}(X)$. If X and Y are SFP-domains, then they are spectral in the Scott topology. Hence, by Corollary 6.2.5, the poset of Scott continuous functions $X \to \mathcal{H}(Y)$ is order isomorphic to the poset of strict and finitely additive functions from the distributive lattice of Scott compact opens $\mathcal{KO}(Y)$ to the lattice of Scott compact opens $\mathcal{KO}(X)$.

Let Y be a metric space taken with the metric topology. By definition, the underlying set of the lower space $\mathcal{P}_l(Y)$ coincides with that of the closed powerdomain $\mathcal{P}_{cl}(Y)$. If X is any discrete metric space, then every function from X to $\mathcal{P}_l(Y)$ is lower semi-continuous. By Theorem 6.2.4, the set of all functions $X \to \mathcal{P}_{cl}(Y)$ is isomorphic to the set of all completely additive functions from the lattice of metric opens $\mathcal{O}(Y)$ to the lattice of metric opens $\mathcal{O}(X)$ (since X is discrete, the latter coincides with the discrete topology on X). In case both X and Y are compact ultra-metric spaces (and hence Stone spaces in their metric topology) we can use the characterization of Corollary 6.2.5. Notice that the lower topology on $\mathcal{P}_{cl}(Y)$ (which is $\mathcal{T}_0$) does not coincide with the metric topology (which is $\mathcal{T}_2$). We need to consider non-symmetric metric spaces. For ω-algebraic complete quasi metric spaces a result which generalizes Proposition 6.2.7 is presented in [35].

Upper state transformers

Next we give a duality between upper state transformers and M-multiplicative predicate transformers. Intuitively, upper state transformers are models for non-deterministic computations of which the outputs 'must' satisfy a given affirmative predicate. For an M-multiplicative predicate transformer π, a state x satisfies $\pi(P)$ if the computation represented by π at input x is guaranteed to terminate in a state satisfying the affirmative predicate P.

According to the informal definition of safety and liveness predicates given in [126], an arbitrary predicate can always be expressed as the intersection of a safety and a liveness predicate. This fact leads [8] to the topological definition of safety predicates as closed subsets, whereas a liveness predicate can be identified with a *dense subset* (the complement does not contain non-empty open sets). It is not hard to see that in any topological space X, any subset of X can be expressed as the intersection of a closed set with a dense one.

Since we are concerned with affirmative and refutative predicates, it is clear that M-multiplicative predicate transformers are not liveness predicate transformers in the sense of [126].

Theorem 6.2.8 *Let X and Y be two topological spaces. The poset of upper state transformers $X \to \mathcal{P}_u(Y)$ is order isomorphic to the poset of M-multiplicative functions $\mathcal{O}(Y) \to_M \mathcal{O}(X)$.*

Proof. The proof is similar to that of Theorem 6.2.4, making use of Proposition 6.1.4. For every continuous function $f : X \to \mathcal{P}_u(Y)$ and M-multiplicative predicate transformer $\pi : \mathcal{O}(Y) \to_M \mathcal{O}(X)$ define the assignments $f \mapsto \omega(f)$ and $\pi \mapsto \omega^{-1}(\pi)$ by

$$\begin{aligned} \omega(f) \quad &= \lambda o \in \mathcal{O}(Y).\{x \in X \mid f(x) \subseteq o\} \qquad \text{and} \\ \omega^{-1}(\pi) &= \lambda x \in X. \bigcap\{o \in \mathcal{O}(Y) \mid x \in \pi(o)\}. \end{aligned}$$

For every open o of Y, $\omega(f)(o) = f^+(o)$. Hence $\omega(f)$ is M-multiplicative and, because f is upper semi-continuous as multifunction, well-defined. To prove that $\omega^{-1}(\pi)$ is well-defined, observe that an arbitrary intersection of open sets is upper closed with respect to the specialization order, and, for every $o \in \mathcal{O}(Y)$,

$$\begin{aligned} (\omega^{-1}(\pi))^+(o) &= \{x \in X \mid \omega^{-1}(\pi)(x) \subseteq o\} \\ &= \{x \in X \mid \bigcap\{o \in \mathcal{O}(Y) \mid x \in \pi(o)\} \subseteq o\} \\ &= \{x \in X \mid x \in \pi(o)\} \quad \text{[Lemma 6.2.2]} \\ &= \pi(o). \end{aligned}$$

Since $\pi(o)$ is open in X, $\omega^{-1}(\pi)$ is upper semi-continuous. Thus it is well-defined. The above also proves that ω^{-1} is a right inverse of ω. It is also a left inverse because, for every $x \in X$,

$$\begin{aligned} \omega^{-1}(\omega(f))(x) &= \bigcap\{o \in \mathcal{O}(Y) \mid x \in \omega(f)(o)\} \\ &= \bigcap\{o \in \mathcal{O}(Y) \mid f(x) \subseteq o\} \\ &= f(x), \end{aligned}$$

where the latter equality follows because $f(x)$ is upper closed in Y, and hence a saturated set. Preservation of orders is immediate. $\square$

As for the case of lower state transformers, if we exclude the empty set as possible output result of an upper state transformer then the corresponding restriction on M-multiplicative predicate transformers is strictness.

For every space X, the underlying set of the upper space $\mathcal{P}_u(X)$ ordered by the specialization preorder is a complete lattice. If X is a $\mathcal{T}_1$ space, then the underlying set of $\mathcal{P}_u(X)$ coincides with the full powerset of X because every set is upper closed.

What restrictions are needed on the underlying space in order that the compact restriction of the upper powerspace is a dcpo or an algebraic dcpo? For a dcpo the question has been answered in [108]: the underlying space should be sober. This is proved using a bijective correspondence between the elements of the compact upper powerspace (compact saturated sets) and the Scott open filters of the lattice of opens sets (for a proof of this statement we refer the reader to Corollary 9.3.11).

Proposition 6.2.9 *Let X be a sober space. If S is an arbitrary collection of compact saturated subsets of X directed with respect to superset inclusion then $\bigcap S$ is also saturated and compact. Moreover, for any open $o \in \mathcal{O}(X)$, if $\bigcap S \subseteq o$ then there exists $q \in S$ such that $q \subseteq o$.* $\square$

The first statement of the above proposition gives soberness as a sufficient condition for the compact upper powerspace to be a dcpo (more generally, Schalk proved that if a space is sober then so is its non-empty compact upper space [170, Lemma 7.20]). The second statement says that compact opens are compact elements for the dcpo $\mathcal{P}_u^{co}(X)$. However this dcpo need not to be algebraic. The algebraicity is obtained by restricting the attention to sober and locally open compact spaces.

Lemma 6.2.10 *Let X be a sober locally open compact space. The underlying set of the compact upper space $\mathcal{P}_u^{co}(X)$ ordered by the specialization order is an algebraic dcpo with as compact elements the compact open sets. Moreover, the Scott topology on $\mathcal{P}_u^{co}(X)$ coincides with the upper topology.*

Proof. We need to prove that every compact saturated set q can be expressed as least upper bound of the compact open sets below q. Because X is locally open compact, every open set can be obtained as a directed union of compact opens. Hence, if q is a compact saturated set such that $q \subseteq o$ for some open set o, then there exists a compact open u such that $q \subseteq u \subseteq o$. For a compact saturated sets q, this implies

$$q = \bigcap\{o \in \mathcal{O}(X) \mid q \subseteq o\} = \bigcap\{u \in \mathcal{KO}(X) \mid q \subseteq u\}.$$

Hence the collection of compact saturated sets is an algebraic dcpo when ordered by superset inclusion.

Next we prove that the Scott topology and the upper topology on $\mathcal{P}_u(X)$ coincide. The upper closure of a compact open o in $\mathcal{P}_u(X)$ is a basic open for the Scott topology, and by definition it coincides with the basic open $U_o = \{q \mid q \subseteq o\}$ of the upper topology. Hence the Scott topology on $\mathcal{P}_u(X)$ is included in the upper topology. Conversely, let $o \in \mathcal{O}(X)$ and consider the basic open set U_o of the upper topology on $\mathcal{P}_u(X)$. It is clearly upper closed, and if $S \subseteq \mathcal{P}_u(X)$ is a directed set such that $\bigcap S \in U_o$ then, by Corollary 6.2.9, there exists $q \in S$ such that $q \subseteq o$. Therefore U_o is Scott open. $\square$

Since algebraic dcpo's taken with the Scott topology are sober, the above lemma implies that the compact upper space of a locally open compact sober space is again sober. In particular, if X is an algebraic cpo, then so is the poset of all Scott compact saturated subsets of X ordered by superset inclusion. The following characterization theorem can be found in [179] for ω-algebraic cpo's, and in [4,146] for the general case.

Proposition 6.2.11 *Let X be an algebraic cpo taken with the Scott topology. The Smyth powerdomain $\mathcal{S}(X)$ together with its Scott topology is isomorphic in* **Sp** *to the non-empty, compact upper powerspace $\mathcal{P}_u^{co}(X) \setminus \{\emptyset\}$.* $\square$

In order to apply the isomorphism of Theorem 6.2.8 to upper state transformers with values in an upper compact subset, we need to find a corresponding restriction on the predicate transformer side. The definition of compact sets as finitarily specifiable theory introduced in Chapter 5 is of help here.

Theorem 6.2.12 *Let X and Y be two topological spaces. The isomorphism of Theorem 6.2.8 restricts to an order isomorphism between the poset of compact upper state transformers $X \to \mathcal{P}_u^{co}(Y)$ and the poset of M-multiplicative and Scott continuous functions $\mathcal{O}(Y) \to_{c,M} \mathcal{O}(X)$.*

Proof. Let $f \in X \to \mathcal{P}_u^{co}(Y)$. Also let $\pi : \mathcal{O}(Y) \to_{c,M} \mathcal{O}(X)$ be a Scott continuous function. We need to prove $\omega(f)$ Scott continuous and $\omega^{-1}(\pi)(x)$ compact for all $x \in X$. Let S be a directed subset of opens in Y.

If $x \in \omega(f)(\bigcup S)$ then $f(x) \subseteq \bigcup S$. By compactness of $f(x)$ it follows that $f(x) \subseteq o$ for some $o \in S$. Therefore $x \in \bigcup\{\omega(f)(o) \mid o \in S\}$. Since $\omega(f)$ is monotone, being M-multiplicative, it follows that $\omega(f)$ is Scott continuous.

Take now $x \in X$ and assume $\omega^{-1}(\pi)(x) \subseteq \bigcup S$. By Lemma 6.2.2 then x is in $\pi(\bigcup S)$. Since π is Scott continuous, there exists $o \in S$ such that $x \in \pi(o)$. Using Lemma 6.2.2 again it follows that $\omega^{-1}(\pi)(x) \subseteq o$, that is $\omega^{-1}(\pi)(x)$ is compact. $\Box$

Using Corollary 9.3.11, Smyth [179] was the first who realized that for a sober space Y, the poset of upper state transformers $X \to \mathcal{P}_u^{co}(Y)$ is order isomorphic to the poset of finitely multiplicative and Scott continuous functions in $\mathcal{O}(Y) \to \mathcal{O}(X)$. In the above theorem, we do not have the requirement of Y being sober, but we consider M-multiplicativity instead of finitely multiplicativity. Hence, if Y is a sober space then a (Scott-)continuous function in $\mathcal{O}(Y) \to \mathcal{O}(X)$ is finite multiplicative if and only if it is M-multiplicative. Best [29] has proved a similar result for countable flat cpo's.

Corollary 6.2.13 *Let X and Y be two sets and let $\pi : \mathcal{P}(Y) \to \mathcal{P}(X)$ Scott continuous. If π preserves binary intersections then it preserves all non-empty intersections.*

Proof. Consider the flat dcpo $Y_\perp$ taken with the Scott topology. Notice that the latter equals $\mathcal{P}(Y) \cup \{Y_\perp\}$. Hence we can extend π to a Scott-continuous function from $\mathcal{O}(Y_\perp) \to \mathcal{P}(X)$ by mapping $\pi(Y_\perp) = X$. If π preserves binary intersections then its extension preserves all finite intersections (being top preserving). Since $Y_\perp$ is a sober space, the extension of π is M-multiplicative. Hence $\pi : \mathcal{P}(Y) \to \mathcal{P}(X)$ preserves all non-empty intersections. $\Box$

Another consequence of the combination of the result of Smyth [179] and Theorem 6.2.12 is the following.

Corollary 6.2.14 *Let X and Y be two spectral spaces. The poset of compact upper state transformers $X \to \mathcal{P}_u^{co}(Y)$ is order isomorphic to the poset of finitely multiplicative functions in $\mathcal{KO}(Y) \to \mathcal{KO}(X)$.*

Proof. Since Y is spectral, it is also sober. Moreover the intersection of compact opens is compact open by definition. Hence every Scott continuous and M-multiplicative function $\pi : \mathcal{O}(Y) \to \mathcal{O}(X)$ restricts to a finite meet preserving function in $\mathcal{KO}(Y) \to \mathcal{KO}(X)$. Conversely, every finite meet preserving function $\pi : \mathcal{KO}(Y) \to \mathcal{KO}(X)$ extends by means of ideal completion uniquely to a Scott continuous and finite meet preserving function in $\mathcal{O}(Y) \to \mathcal{O}(X)$ the restriction of which to compact sets is exactly π. Since Y is sober, this extension of π is M-multiplicative. $\square$

Let X and Y be two algebraic cpo's. From Theorem 6.2.8 and Proposition 6.2.11 we have that the poset of Scott continuous functions $X \to \mathcal{S}(Y)$ is order isomorphic to the poset of strict, Scott continuous and finite multiplicative functions from the lattice of Scott opens $\mathcal{O}(Y)$ to the lattice of Scott opens $\mathcal{O}(X)$. Moreover, if X and Y are SFP-domains, then they are spectral in the Scott topology. Hence, by Corollary 6.2.14, the poset of Scott continuous functions from $X \to \mathcal{S}(Y)$ is order isomorphic to the poset of strict and finitely multiplicative functions from the distributive lattice of Scott compact opens $\mathcal{KO}(Y)$ to the lattice of Scott compact opens $\mathcal{KO}(X)$.

Let X be a discrete metric space and let Y be a metric space. Thus every function from X to $\mathcal{P}_u^{co}(Y)$ is continuous. Notice that the underlying sets of $\mathcal{P}_u^{co}(Y)$ and of the metric compact powerdomain $\mathcal{P}_{co}(Y)$ coincide. Therefore, by Theorem 6.2.12, the set of all functions $X \to \mathcal{P}_{co}(Y)$ is isomorphic to the set of all Scott continuous and finitely multiplicative functions from the lattice of metric opens $\mathcal{O}(Y)$ to the lattice of metric opens $\mathcal{O}(X)$. In case both X and Y are compact ultra-metric spaces, the set $X \to \mathcal{P}_{co}(Y)$ is isomorphic to the set of strict and finitely multiplicative functions from the distributive lattice of metric compact opens $\mathcal{KO}(Y)$ to the lattice of metric compact opens $\mathcal{KO}(X)$.

6.3 Pairs of predicate transformers

In Chapter 3 we have seen that the Egli-Milner state transformers are dual to the Nelson predicate transformers. The natural topological generalization of the Egli-Milner state transformers are the convex state transformers. In order to generalize the Nelson predicate transformers we need to consider pairs $\langle \pi, \rho \rangle$ of topological predicate transformers, where π is M-multiplicative and ρ is completely additive. In this way we can model both the positive and the negative information about a computation. However, we have to restrict our considerations to those pairs $\langle \pi, \rho \rangle$ of predicate transformers which represent the same computation. What we need is a stability lemma similar to Lemma 6.2.2 and Lemma 6.2.3. The definition below is inspired by the work of Johnstone [113] on the Vietoris powerlocale.

Definition 6.3.1 *Given two spaces X and Y, a pair $\langle \pi, \rho \rangle$ of functions from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$ is said to be* jointly multiplicative *if π is M-multiplicative, ρ is completely additive, and*

(i) $\pi(o_1 \cup o_2) \subseteq \pi(o_1) \cup \rho(o_2)$, *and*

(ii) $(\bigcap S \cap o_1) \subseteq o_2$ *implies* $(\bigcap\{\pi(o) \mid o \in S\} \cap \rho(o_1)) \subseteq \rho(o_2)$.

for opens o_1, o_2 of Y and $S \subseteq \mathcal{O}(Y)$. Jointly multiplicative functions are ordered componentwise by the extension to functions of subset inclusion.

For a pair $\langle \pi, \rho \rangle$ of jointly multiplicative functions, according to the above definition there are two 'non-observable' requirements (in the sense that they involve sets which need not to be open): the M-multiplicativity of π and the second condition of joint multiplicativity. In the previous section we have shown that if π is Scott continuous and the space Y is sober, then M-multiplicativity is equivalent to finite multiplicativity. The latter is clearly an observable and finitary requirement.

The non-observability of condition (ii) of Definition 6.3.1 is more delicate. In locale theory, for the construction of the Vietoris powerlocale the following condition is required [113] instead of (ii),

(6.1)$\pi(o_1) \cap \rho(o_2) \subseteq \rho(o_1 \cap o_2)$

for all $o_1, o_2 \in \mathcal{O}(Y)$. Notice that (i) of Definition 6.3.1 and the above (6.1) are the modal axioms relating the $\Box$ and $\Diamond$ operators in negation free modal logic (often called Hennessy-Milner logic) [93].

For all spaces X and Y, if $\langle \pi, \rho \rangle$ is a jointly multiplicative pair of functions from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$ then (6.1) clearly holds. The converse holds if we restrict Y to be a coherent space and π to be Scott continuous.

Lemma 6.3.2 *Let X and Y be two spaces such that Y is coherent. For every pair $\langle \pi, \rho \rangle$ of Scott continuous functions from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$ such that π preserves finite meets, and ρ preserves finite joins, the following two statements are equivalent:*

(i) $\pi(o_1) \cap \rho(o_2) \subseteq \rho(o_1 \cap o_2)$ *for all* $o_1, o_2 \in \mathcal{O}(Y)$;

(ii) $(\bigcap S \cap o_1) \subseteq o_2$ *implies* $(\bigcap\{\pi(o) \mid o \in S\} \cap \rho(o_1)) \subseteq \rho(o_2)$ *for all* $o_1, o_2 \in \mathcal{O}(Y)$ *and* $S \subseteq \mathcal{O}(Y)$.

Moreover, if Y is spectral then both (i) *and* (ii) *are equivalent to*

(iii) $\pi(o_1) \cap \rho(o_2) \subseteq \rho(o_1 \cap o_2)$ *for all* $o_1, o_2 \in \mathcal{KO}(Y)$.

Proof. Obviously (ii) implies (i). Hence we concentrate on the opposite direction. Assume $\pi(o) \cap \rho(o') \subseteq \rho(o \cap o')$ for all opens o and o' of Y. Let $S \subseteq \mathcal{O}(Y)$ and $o_1, o_2 \in \mathcal{O}(Y)$. Because Y is coherent, every open set o of Y is the union of all the compact saturated subsets q of Y such that there exists $u \in \mathcal{O}(Y)$ with $u \subseteq q \subseteq o$. Hence the set $\bigcap S \cap o_1$ is equivalent to the set

$$\bigcap\{q \in \mathcal{KQ}(Y) \mid \exists u \in \mathcal{O}(Y) \colon \exists o \in S \cup \{o_1\} \colon u \subseteq q \subseteq o\}.$$

Next, we use the fact that Y is sober and that compact saturated sets are closed under finite intersections to reformulate Proposition 6.2.9 as follows. Whenever the intersection of compact saturated sets is contained in an open set then the same is true for an intersection of finitely many of them. This fact justifies that $(\bigcap S \cap o_1) \subseteq o_2$ implies that there exist finitely many compact saturated sets $q_1, ... q_n$ such that $q_1 \cap ... \cap q_n \subseteq o_2$, with $u_i \subseteq q_i \subseteq o_i$ for some $o_i \in S \cup \{o_1\}$ and open $u_i \in \mathcal{O}(Y)$. Hence $(q_1 \cap ... \cap q_n) \cap o_1 \subseteq o_2$, where, without loss of generality, we can assume, for all $1 \leq i \leq n$, $u_i \subseteq q_i \subseteq o_i$ for some $o_i \in S$ and $u_i \in \mathcal{O}(Y)$.

Let $u = u_1 \cap ... \cap u_n$. Since $u_1, .., u_n$ are finitely many open sets, u is also an open set. Moreover $u \cap o_1 \subseteq o_2$ because $u \subseteq q_1 \cap ... \cap q_n$. By our assumption on the pair $\langle \pi, \rho \rangle$, $\pi(u) \cap \rho(o_1) \subseteq \rho(u \cap o_1)$. But ρ is monotone and $u \cap o_1 \subseteq o_2$. Thus $\pi(u) \cap \rho(o_1) \subseteq \rho(o_2)$. Notice that $\bigcap S \subseteq u$ because, for all $1 \leq i \leq n$, $o_i \in S$ and

$$o_i = \bigcup\{u \in \mathcal{O}(Y) \mid \exists q \in \mathcal{KQ}(Y) \colon u \subseteq q \subseteq o_i\}$$

as Y is coherent. Thus $\bigcap\{\pi(o) \mid o \in S\} \subseteq \pi(u)$, from which follows that $(\bigcap\{\pi(o) \mid o \in S\} \cap \rho(o_1)) \subseteq \rho(o_2)$.

Assume now Y is a spectral space. We prove that (iii) implies (i). The other direction follows immediately.

Let o_1 and o_2 be two open sets of Y. Because Y is spectral they can be written as directed union of all compact open sets below them. Below, let u and v range over compact open sets. Because π and ρ are both Scott continuous, we have

$$\begin{aligned}
\pi(o_1) \cap \rho(o_2) &= \pi(\bigcup\{u \mid u \subseteq o_1\}) \cap \rho(\bigcup\{v \mid v \subseteq o_2\}) \\
&= \bigcup\{\pi(u) \mid u \subseteq o_1\} \cap \bigcup\{\rho(v) \mid v \subseteq o_2\} \\
&= \bigcup\{\pi(u) \cap \rho(v) \mid u \subseteq o_1 \ \&\ v \subseteq o_2\} \\
&\subseteq \bigcup\{\rho(u \cap v) \mid u \cap v \subseteq o_1 \cap o_2\} \quad \text{[by (iii)]} \\
&= \rho(o_1 \cap o_2).
\end{aligned}$$

Since spectral spaces are coherent, (i) is equivalent to (ii). Hence (iii) implies both (i) and (ii). □

As a consequence, if Y is a coherent space then the jointly multiplicative and Scott continuous predicate transformers $\langle \pi, \rho \rangle$ from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$ can be described using only open sets, substituting finite multiplicativity for M-multiplicativity, and condition (ii) of Definition 6.3.1 by the equivalent condition (6.1).

Lemma 6.3.3 *Let X and Y be two spaces such that Y is coherent. The poset of all jointly multiplicative and Scott continuous predicate transformers from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$ is a cpo.*

Proof. Let $D = \{\langle \pi_i, \rho_i \rangle \mid i \in I\}$ be a directed set of jointly multiplicative and Scott continuous functions from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$. Define $\pi(o) = \bigcup_I \pi_i(o)$ and $\rho(o) = \bigcup_I \rho_i(o)$ for every open o of Y. By Proposition 6.2.9 and Theorem 6.2.12 the function π is M-multiplicative and Scott continuous. Thus π is the least upper bound of all π_i's. The function ρ is completely additive by definition, and hence is the least upper bound of all ρ_i's. We need to prove that $\langle \pi, \rho \rangle$ is a jointly multiplicative pair.

Let o_1 and o_2 be two open sets of Y. If $x \in \pi(o_1 \cup o_2)$ then there exists $k \in I$ such that $x \in \pi_k(o_1 \cup o_2)$. Since $\langle \pi_k, \rho_k \rangle$ is jointly multiplicative, $\pi_k(o_1 \cup o_2) \subseteq \pi_k(o_1) \cup \rho_k(o_2)$. But $\pi_k(o_1) \subseteq \pi(o_1)$ and $\rho_k(o_2) \subseteq \rho(o_2)$. Thus $x \in \pi(o_1) \cup \rho(o_2)$, that is, condition (i) of Definition 6.3.1 holds.

Because Y is a coherent space, by Lemma 6.3.2, condition (ii) of Definition 6.3.1 is equivalent to the finitary condition (6.1). Assume o_1 and o_2 are two open sets of Y and let $x \in \pi(o_1) \cap \rho(o_2)$. By directness of the set D and the definitions of π and ρ, there exists $k \in I$ such that $x \in \pi_k(o_1)$ and $x \in \rho_k(o_2)$. Since $\langle \pi_k, \rho_k \rangle$ is jointly multiplicative, $\pi_k(o_1) \cap \rho_k(o_2) \subseteq \rho_k(o_1 \cap o_2)$. Thus $x \in \rho_k(o_1 \cap o_2) \subseteq \rho(o_1 \cap o_2)$.

Therefore $\langle \pi, \rho \rangle$ is jointly multiplicative and the poset of all jointly multiplicative and Scott continuous predicate transformers from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$ is a dcpo. The pair of functions mapping every open set of Y to the empty set is jointly multiplicative and Scott continuous. Hence they form the bottom element of the dcpo of jointly multiplicative and Scott continuous predicate transformers. $\Box$

We are interested in jointly multiplicative predicate transformers because they represent the positive and the negative information of the same computation, as formally stated in the following stability lemma.

Lemma 6.3.4 *Let X and Y be two spaces and $\langle \pi, \rho \rangle$ be a a pair of jointly multiplicative functions from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$. For $x \in X$, let us denote by $q(x, \pi)$ the set $\bigcap\{o \in \mathcal{O}(Y) \mid x \in \pi(o)\}$ and by $o(x, \rho)$ the set $\bigcup\{o \in \mathcal{O}(Y) \mid x \notin \rho(o)\}$. For every $u \in \mathcal{O}(Y)$ we have*

(i) $x \in \pi(u)$ *if and only if* $q(x, \pi) \cap (Y \setminus o(x, \rho)) \subseteq u$,

(ii) $x \notin \rho(u)$ *if and only if* $q(x, \pi) \cap (Y \setminus o(x, \rho)) \subseteq Y \setminus u$.

Proof. (i) The direction from left to right is trivial and hence omitted. Assume $q(x, \pi) \cap (Y \setminus o(x, \rho)) \subseteq u$. Then $q(x, \pi) \subseteq u \cup o(x, \rho)$. Since $o(x, \rho)$ is open (being union of opens) and π is M-multiplicative, $x \in \pi(u \cup o(x, \rho))$ by Lemma 6.2.2. But $\pi(u \cup o(x, \rho)) \subseteq \pi(o) \cup \rho(o(x, \rho))$ because π and ρ are jointly multiplicative. Since ρ is completely additive we have that $x \notin \rho(o(x, \rho))$ by definition of $o(x, \rho)$. Therefore $x \in \pi(u)$.

(ii) As above we omitt the direction from left to right because is trivial. Assume $q(x, \pi) \cap (Y \setminus o(x, \rho)) \subseteq Y \setminus u$. Then $q(x, \pi) \cap u \subseteq o(x, \rho)$, which implies

$$(\bigcap\{\pi(o) \mid x \in \pi(o)\} \cap \rho(u)) \subseteq \rho(o(x, \rho))$$

because $\langle \pi, \rho \rangle$ is jointly multiplicative. Since ρ is completely additive, x is not in $\rho(o(x, \rho))$. But $x \in \bigcap\{\pi(o) \mid x \in \pi(o)\}$, therefore $x \notin \rho(u)$. $\Box$

Next we use the above stability lemma to relate jointly multiplicative predicate transformers and convex state transformers by an isomorphism that generalizes the result in Chapter 3 for Nelson predicate transformers.

Theorem 6.3.5 *Let X and Y be two spaces. The poset of convex state trans-*

formers $X \to \mathcal{P}_c(Y)$ is order isomorphic to the poset of all jointly multiplicative pairs of predicate transformers in $\mathcal{O}(Y) \to \mathcal{O}(X)$. Also, the above isomorphism cuts down to an order isomorphism between compact and convex state transformers $X \to \mathcal{P}_c^{co}(Y)$ and the poset of all pairs of Scott continuous functions in $\mathcal{O}(Y) \to \mathcal{O}(X)$ which are jointly multiplicative.

Proof. For a convex state transformer $f : X \to \mathcal{P}_c(Y)$ define $\eta(f)$ to be the pair of functions from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$

$$\omega(f) = \lambda o \in \mathcal{O}(Y).\{x \mid f(x) \subseteq o\} \qquad \text{and}$$
$$\gamma(f) = \lambda o \in \mathcal{O}(Y).\{x \mid f(x) \cap o \neq \emptyset\}.$$

Since f is continuous as a multifunction, both functions above are well-defined. Moreover, $\omega(f)$ is M-multiplicative and $\gamma(f)$ is completely additive. Next we prove they are jointly multiplicative.

Let o_1 and o_2 be two open subsets of Y. If $x \in \omega(f)(o_1 \cup o_2)$ then $f(x) \subseteq o_1 \cup o_2$. Towards a contradiction, assume both $f(x) \not\subseteq o_1$ and $f(x) \cap o_2 = \emptyset$. Then $f(x) \not\subseteq o_1 \cup o_2$, hence the contradiction. Thus $f(x) \subseteq o_1$ or $f(x) \cap o_2 \neq \emptyset$, that is, $x \in \omega(f)(o_1) \cup \gamma(f)(o_2)$.

Let $S \subseteq \mathcal{O}(Y)$ and let o_1, o_2 be two open subsets of Y such that $\bigcap S \cap o_1 \subseteq o_2$. If $x \in \omega(f)(o)$ for all $o \in S$ and $x \in \gamma(f)(o_1)$, then $f(x) \subseteq \bigcap S$ and $f(x) \cap o_1 \neq \emptyset$. Hence there exists $y \in f(x)$ such that $y \in \bigcap S \cap o_1 \subseteq o_2$. It follows that $f(x) \cap o_2 \neq \emptyset$, and hence $x \in \gamma(f)(o_2)$. Therefore the pair $\eta(f) = \langle \omega(f), \gamma(f) \rangle$ is jointly multiplicative.

Consider now the pair $\langle \pi, \rho \rangle$ of jointly multiplicative predicate transformers in $\mathcal{O}(Y) \to \mathcal{O}(X)$. Define $\eta^{-1}(\langle \pi, \rho \rangle)(x)$, for every $x \in X$, by

$$\bigcap \{o \in \mathcal{O}(Y) \mid x \in \pi(o)\} \cap (Y \setminus \bigcup \{o \in \mathcal{O}(Y) \mid x \notin \rho(o)\}). \tag{6.2}$$

We prove that $\eta^{-1}(\langle \pi, \rho \rangle)(x)$ is convex closed. Let cl be the closure operator induced by the topology $\mathcal{O}(Y)$. Since $Y \setminus \bigcup \{o \in \mathcal{O}(Y) \mid x \notin \rho(o)\}$ is a closed set,

$$cl(\eta^{-1}(\langle \pi, \rho \rangle)(x)) \subseteq Y \setminus \bigcup \{o \in \mathcal{O}(Y) \mid x \notin \rho(o)\}.$$

Similarly, the upper closure $\uparrow \eta^{-1}(\langle \pi, \rho \rangle)(x)$, with respect to the order induced by $\mathcal{O}(Y)$, is included in the saturated set $\bigcap\{o \in \mathcal{O}(Y) \mid x \in \pi(o)\}$. Hence the convex closure of $\eta^{-1}(\langle\pi,\rho\rangle)(x)$ is included in (6.2). Since the other direction is trivial, $\eta^{-1}(\langle\pi,\rho\rangle)(x)$ is convex closed.

Next we prove that $\eta^{-1}(\langle\pi,\rho\rangle)$ is both upper and lower semi-continuous. For every $o \in \mathcal{O}(Y)$ we have

$$\begin{aligned}\eta^{-1}(\langle\pi,\rho\rangle)^{+}(o) &= \{x \in X \mid \eta^{-1}(\langle\pi,\rho\rangle)(x) \subseteq o\}\\ &= \{x \in X \mid x \in \pi(o)\} \quad \text{[Lemma 6.3.4]}\\ &= \pi(o),\end{aligned}$$

and also

$$\begin{aligned}\eta^{-1}(\langle\pi,\rho\rangle)^{-}(o) &= \{x \in X \mid \eta^{-1}(\langle\pi,\rho\rangle)(x) \cap o \neq \emptyset\}\\ &= \{x \in X \mid \eta^{-1}(\langle\pi,\rho\rangle)(x) \not\subseteq Y \setminus o\}\\ &= \{x \in X \mid x \in \rho(o)\} \quad \text{[Lemma 6.3.4]}\\ &= \rho(o).\end{aligned}$$

This proves not only that $\eta^{-1}(\langle\pi,\rho\rangle)$ is a convex state transformer, but also that $\eta^{-1}(\langle\pi,\rho\rangle)$ is a right inverse of η. It not hard to see that it is also a left inverse by combining Theorem 6.2.4 and Theorem 6.2.8.

Further, η and η^{-1} are both monotone due to Lemma 6.1.5 and Theorems 6.2.4 and 6.2.8.

By Theorem 6.2.12 and because the intersection of a compact set with a closed one gives again a compact set, it follows that the isomorphism (η,η^{-1}) cuts down to an order-isomorphism between the poset of compact and convex state transformers, and the poset of all pairs of Scott continuous functions in $\mathcal{O}(Y) \to \mathcal{O}(X)$ which are jointly multiplicative. $\square$

As for the cases of the upper space and of the lower space, the above isomorphisms cuts down to a finitary isomorphism if we consider spectral spaces.

Corollary 6.3.6 *Let X and Y be two spectral spaces. The poset of compact convex state transformers $X \to \mathcal{P}_c^{co}(Y)$ is order isomorphic to the poset of all pairs $\langle\pi,\rho\rangle$ of functions in $\mathcal{KO}(Y) \to \mathcal{KO}(X)$ such that*

(i) *π is finitely multiplicative;*
(ii) *ρ is finitely additive;*

(iii) $\pi(o_1 \cup o_2) \subseteq \pi(o_1) \cup \rho(o_2)$ *for all* $o_1, o_2 \in \mathcal{KO}(Y)$*;*

(iv) $\pi(o_1) \cap \rho(o_2) \subseteq \rho(o_1 \cap o_2)$ *for all* $o_1, o_2 \in \mathcal{KO}(Y)$*.*

Proof. Immediate from Corollaries 6.2.5 and 6.2.14, and Lemma 6.3.2. □

Despite the mathematical elegance of the presentation of the convex space, it does not have many of the closure properties which the other power constructions enjoy. In general, the underlying set of $\mathcal{P}_c^{co}(X)$ taken with the order induced by the Vietoris topology, is not a complete lattice nor a dcpo even if X is an algebraic dcpo with the Scott topology [4, Exercise 11.(e)]. As a consequence, neither sober spaces nor sober and locally open compact spaces are closed under the compact convex space construction. Using the above isomorphism and Lemma 6.3.3 we obtain an easy proof that $\mathcal{P}_c^{co}(X)$ is a cpo whenever X is a coherent space. The general situation, i.e. to find a topological characterization of the Plotkin powerdomain, seems to be hopeless. The following characterization theorems can be found in [179] and [4,146].

Proposition 6.3.7 *Let X be an ω-algebraic cpo taken with the Scott topology. The Plotkin powerdomain $\mathcal{E}(X)$ together with its Scott topology is isomorphic in* **Sp** *to the non-empty, compact convex space $\mathcal{P}_c^{co}(X) \setminus \{\emptyset\}$. The same holds if X is an algebraic cpo such that, when taken with the Scott topology, it forms a coherent space.* □

From the above proposition and Theorem 6.3.5, we have, for ω-algebraic cpo's X and Y, that the poset of Scott continuous functions $X \to \mathcal{E}(Y)$ is order isomorphic to the poset of all pairs of strict and Scott continuous functions from the lattice of Scott opens $\mathcal{O}(Y)$ to the lattice of Scott opens $\mathcal{O}(X)$ which are jointly multiplicative. If X and Y are SFP domains, then we can apply Corollary 6.3.6 to obtain a finitary duality.

For metric spaces we have the following characterization result [141].

Proposition 6.3.8 *Let X be a metric space taken with the metric topology. The metric compact powerdomain $\mathcal{P}_{co}(X)$ together with the metric topology coincides with the compact convex space $\mathcal{P}_c^{co}(X)$* □

The above proposition can be applied as follows. If X and Y are two metric spaces, by Theorem 6.3.5, the set of all metric continuous functions $X \to \mathcal{P}_{co}(Y)$ (seen as a discrete poset) is order-isomorphic to the poset of all pairs of Scott continuous functions from the lattice of metric opens $\mathcal{O}(Y)$ to the lattice of metric opens $\mathcal{O}(X)$ which are jointly multiplicative. If X and Y are compact ultra-metric spaces, then in their metric topologies they are Stone spaces. Hence we can apply Corollary 6.3.6 to obtain a finitary duality.

It is easy to see that if X is a set then the set of all finite subsets of X (taken with the discrete topology) coincides with the compact convex powerspace of X. Again, we can apply Theorem 6.3.5 to describe it by jointly multiplicative functions.

6.4 Concluding notes

Dualities for the convex powerspace provide a natural setting for negation-free modal logics (also called Hennessy-Milner logics). Our approach differs from the one taken by Goldblatt [80] and Abramsky [3] because our axioms relating the $\Box$ operator with the $\Diamond$ operator hold also in an infinitary setting. It is an important topic for further investigation to define an infinitary Hennessy-Milner logic for the convex powerspace.

The results in this chapter are in the concrete framework where predicate transformers are functions between collections of open sets. More abstractly, we could have used frames to represent abstract collections of affirmative predicates by restricting our attention to sober spaces (for the results of last section coherent spaces would be necessary). The duality between frames and sober spaces [112] could then be used to reconstruct points from frames. In Chapter 8 we discuss an abstract algebraic representation of $\mathcal{T}_0$ spaces. All results in this section can be easily adapted to this algebraic framework.

To fully generalize the results of Part I, it remains a challenge to define a 'meaningful' notion of topological state transformers which are dual to the monotonic (or perhaps Scott continuous) functions between lattices of affirmative predicates. More speculatively, for algebraic cpo's the duality of Chapter 4 seems to suggest the composition of the Smyth with the Hoare powerdomain (or vice-versa, since they commute [89,132]). This is correct in the localic case: the lower and upper powerlocales commute, and maps from X to $\mathcal{P}_u(\mathcal{P}_l(Y))$ are equivalent to Scott continuous functions from $\Omega(Y)$ to $\Omega(X)$ [115].